CURRENT DISTRIBUTION IN A FLAT MAGNETOHYDRODYNAMIC CHANNEL FOR THE MOTION OF AN ELECTRICALLY CONDUCTING MEDIUM IN A STRONG MAGNETIC FIELD

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In a strong magnetic field the ions as well as the electrons acquire Larmor rotation. In this case the electric field in the channel and its integral characteristics depend both on the geometry of the channel and the external magnetic field strength, as well as on the physical state and chemical composition of the moving medium. Phenomena occur which are observed in a channel when only the spiral motion of the electrons is taken into account (current concentration on the ends of the electrodes, distortion of the streamlines in the center of the channel etc.).

However, the presence of ion "slipping" exerts a much more marked effect on the integral characteristics of channels. Thus, for example, the power of a magnetohydrodynamic generator in a strong magnetic field becomes constant and remains so as the field increases still further. This property of an energy converter is explained by the fact that the internal resistance of the generator has a square law dependence on the magnetic field.

Another important phenomenon is connected with the "slipping" of ions. In a strong magnetic field the Hall electromotive force disappears, the electrical conductivity of the medium becomes a scalar instead of being a tensor, and the current distribution pattern in the channel assumes the same form as in the case in which there is no Larmor rotation of the electrons. The electrical conductivity of the medium is then taken to mean its effective value which is a function of the magnetic field. In the present paper the task of finding the current distribution in a channel is reduced to solving a boundary value problem for a special class of periodic functions. For this we use the theory of boundary-value problems for the class of automorphic functions.

§1. In the solution of the problem we assume that the magnetic Reynolds number R_m is much less than unity, so that the induced magnetic field is not taken into account. The external magnetic field $H(0, 0, H_z)$ in the channel $-\infty < x < \infty$, $0 \le y \le h$ of the magnetohydrodynamic generator with symmetrically placed electrodes (Fig. 1) is taken to be uniform and in a direction normal to the flow of the medium $\mathbf{v}(u(x, y), v(x,y),0)$.

In a strong magnetic field Ohm's law is written in the form [1]

$$\mathbf{j} = \sigma\Big(-\nabla\varphi + \frac{1}{c}\mathbf{v}\times\mathbf{H}\Big) - \frac{\omega_e\tau_e}{H}\mathbf{j}\times\mathbf{H} + \frac{\omega_e\tau_e\omega_i\tau_i}{H^2}(\mathbf{j}\times\mathbf{H})\times\mathbf{H} \quad (H = |\mathbf{H}|). \tag{1.1}$$

Here j is the current density vector, $\varphi(x,y)$ is the electrostatic potential, H is the magnetic field strength vector, σ is the electrical conductivity of the medium in the channel, ω_e and ω_i are the cyclotron frequencies of the electrons and ions, respectively, τ_e^{-1} and τ_i^{-1} are the effective collision frequencies of the electrons and ions, σ, $\omega_e\tau_e$, and $\omega_i\tau_i$ are constants.

Projecting components of (1.1) onto the coordinate axes and solving the resulting system of equations for the current components j_x and j_y, we have after some reduction

$$j_x(x, y) = \frac{\sigma(1+\omega_e\tau_e\omega_i\tau_i)}{(1+\omega_e\tau_e\omega_i\tau_i)^2+\omega_e^2\tau_e^2}\Big[-\frac{\partial\varphi}{\partial x} + \frac{1}{c}vH + \frac{\omega_e\tau_e}{1+\omega_e\tau_e\omega_i\tau_i}\Big(\frac{\partial\varphi}{\partial y} + \frac{1}{c}uH\Big)\Big],$$

$$j_y(x, y) = \frac{\sigma(1+\omega_e\tau_e\omega_i\tau_i)}{(1+\omega_e\tau_e\omega_i\tau_i)^2+\omega_e^2\tau_e^2}\Big[-\frac{\partial\varphi}{\partial y} - \frac{1}{c}uH + \frac{\omega_e\tau_e}{1+\omega_e\tau_e\omega_i\tau_i}\Big(-\frac{\partial\varphi}{\partial x} + \frac{1}{c}vH\Big)\Big]. \tag{1.2}$$

Considering Eq. (1.2) together with the condition for current continuity, for the assumptions outlined above, we have

$$\partial j_x/\partial x + \partial j_y/\partial y = 0 \tag{1.3}$$

and representing the velocity in the form

$$u = \partial\psi/\partial y, \qquad v = -\partial\psi/\partial x, \tag{1.4}$$

we obtain the harmonic function

$$\Delta\chi(x, y) = \Delta\Big(\varphi + \frac{1}{c}\psi H\Big) = 0 \quad \Big(\Delta \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\Big). \tag{1.5}$$

The velocity can always be represented in the form of (1.4) when the medium in the channel can be regarded as incompressible ($\operatorname{div}\mathbf{v} = 0$ and then $\psi(x,y)$ is the stream function), or when the flow is rectilinear $\mathbf{v}(u(y), 0, 0)$, in which case

$$\psi(y) = \int_0^y u(y)\,dy.$$

The components j_x and j_y of the electric current-density vector are expressed in terms of the function $\chi(x,y)$

$$j_x(x, y) = \frac{\sigma(1+\omega_e\tau_e\omega_i\tau_i)}{(1+\omega_e\tau_e\omega_i\tau_i)^2+\omega_e^2\tau_e^2}\Big(-\frac{\partial\chi}{\partial x} + \frac{\omega_e\tau_e}{1+\omega_e\tau_e\omega_i\tau_i}\frac{\partial\chi}{\partial y}\Big),$$

$$j_y(x, y) = \frac{\sigma(1+\omega_e\tau_e\omega_i\tau_i)}{(1+\omega_e\tau_e\omega_i\tau_i)^2+\omega_e^2\tau_e^2}\Big(-\frac{\partial\chi}{\partial y} - \frac{\omega_e\tau_e}{1+\omega_e\tau_e\omega_i\tau_i}\frac{\partial\chi}{\partial x}\Big). \tag{1.6}$$

Thus the problem of finding the current in the channel is reduced to determining the function $\chi(x, y)$.

Let us now formulate the boundary conditions. The channel walls are taken to be impenetrable and made of perfect materials (ideally conducting electrodes and ideal insulators). The normal component of the current is equal to zero at the nonconducting walls, and the tangential component of the electric field is equal to zero at the conducting walls; moreover, the

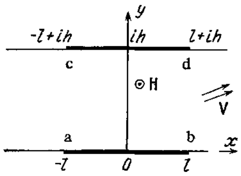

Fig. 1

hydrodynamic condition v = 0 is satisfied at the channel walls which are both impenetrable and motionless. Thus at the insulators

$$j_y = 0 \quad \text{or} \quad \frac{\partial \chi}{\partial y} + \frac{\omega_e \tau_e}{1 + \omega_e \tau_e \omega_i \tau_i} \frac{\partial \chi}{\partial x} = 0 \; ; \tag{1.7}$$

at the electrodes

$$\mathbf{v} = 0, \qquad \partial \varphi / \partial x = 0 \quad \text{or} \quad \partial \chi / \partial x = 0 \; ; \tag{1.8}$$

and at infinity

$$\mathbf{j} = 0 \, . \tag{1.9}$$

To solve problem (1.5), (1.7)–(1.9), the theory of the Riemann boundary problem for automorphic functions is employed.

We now introduce the function

$$W(z) = U(x, y) + iV(x, y) =$$
$$= \partial \chi / \partial x - i \partial \chi / \partial y \quad (z = x + iy) \, , \tag{1.10}$$

which is analytic within the strip $0 \le \operatorname{Im} z \le h$, and which makes it possible for the initial directional derivative problem to be reduced to a Riemann-Hilbert

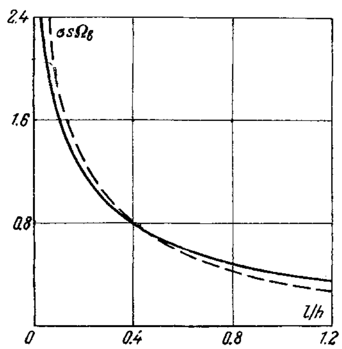

Fig. 2

boundary problem with discontinuous coefficients for the strip.

We substitute the variables

$$\zeta = \pi z / h \qquad (z = x + iy, \;\; \zeta = \xi + i\eta) \, , \tag{1.11}$$

where h is the channel width. Substitution (1.11) changes the geometric scale in the complex-variable plane and as a result we obtain a strip of width π. The end points of the electrodes a, b, c, and d, respectively, become points with the coordinates

$$A = -\pi l/h, \quad B = \pi l/h, \quad C = -\pi l/h + i\pi,$$
$$D = \pi l/h + i\pi \, . \tag{1.12}$$

For the function

$$W_1(\zeta) = W(z) = U_1(\xi, \eta) + iV_1(\xi, \eta) \, , \tag{1.13}$$

from boundary conditions (1.7)–(1.9) we obtain the following Riemann-Hilbert boundary problem in the strip $0 \le \operatorname{Im} \zeta \le \pi$ (subscripts are omitted in what follows):

$$U(t) = 0 \text{ at } L', \;\; V(t) - \frac{\omega_e \tau_e}{1 + \omega_e \tau_e \omega_i \tau_i} U(t) = 0 \text{ at } L'',$$
$$W(\zeta) \to 0 \quad \text{as} \quad |\zeta| \to \infty \, . \tag{1.14}$$

Here L' denotes the segments AB and CD, and L" denotes the remaining portion of the straight lines $\eta = 0$ and $\eta = \pi$; the positive direction for encircling the strip is taken so that the interior of the strip $0 < \operatorname{Im} \zeta < \pi$ remains always on the left.

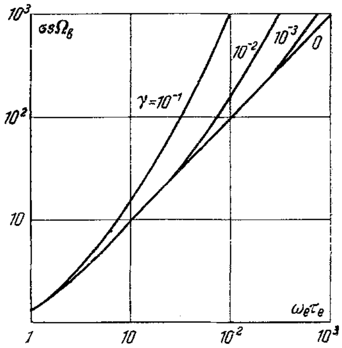

Fig. 3

Problem (1.14) is reduced to a Riemann boundary-value problem for singly periodic functions of period 2π. This is a particular case of the Riemann boundary-value problem for automorphic functions [2,3].

Formulation of the Riemann boundary-value problem involves the introduction of the function $\overline{W}(\bar{\zeta})$ which is the complex conjugate of the function $W(\bar{\zeta})$ at points where the values of $\bar{\zeta}$ are complex conjugates. The strip $-\pi \le \operatorname{Im} \zeta \le \pi$ is taken to be a fundamental region of the zeroth kind for the singly periodic group generated by the transformation $\mu_k(\zeta) = \zeta + 2\pi i k$. The fundamental prime invariant of the group is the function $\exp \zeta$, which is bounded at the left end of the strip and has a pole of the first order at the other end of the strip.

The Riemann problem corresponding to the Riemann-Hilbert problem (1.14) has the form

$$\Psi^+(t) = -\Psi^-(t) \quad \text{at } L'$$
$$(t \in L' + L'') \, ,$$
$$\Psi^+(t) = \frac{1 + \omega_e \tau_e \omega_i \tau_i + i \omega_e \tau_e}{1 + \omega_e \tau_e \omega_i \tau_i - i \omega_e \tau_e} \Psi^-(t) \quad \text{at } L''$$

$$\Psi(\zeta) = \begin{cases} \Psi^+(\zeta) & \text{for} \quad 0 < \operatorname{Im} \zeta < \pi, \;\; \Psi^+(\zeta) = W(\zeta) \, , \\ \Psi^-(\zeta) & \text{for} \; -\pi < \operatorname{Im} \zeta < 0, \;\; \Psi^-(\zeta) = \overline{W(\bar{\zeta})} \, , \end{cases}$$
$$\lim \Psi(\zeta) = 0 \quad \text{as} \quad |\zeta| \to \infty \, . \tag{1.15}$$

In accordance with the physical conditions for the concentration of current near the ends of the electrodes, the solution of problem (1.15) is constructed as a class of functions with integrable singularities at points A, B, C, and D and bounded at the ends of the strip

$$\Psi(\zeta) = (-1)^{\varepsilon} C_1 e^{\zeta} [(e^{\zeta} - e^{A})(e^{\zeta} - e^{D})]^{-1/2 + \varepsilon} \times$$
$$\times [(e^{\zeta} - e^{B})(e^{\zeta} - e^{C})]^{-1/2 - \varepsilon}, \tag{1.16}$$

$$\varepsilon = \frac{1}{\pi} \operatorname{arctg} \frac{\omega_e \tau_e}{1 + \omega_e \tau_e \omega_i \tau_i} \qquad \left(0 \leqslant \varepsilon < \frac{1}{2}\right), \qquad (1.17)$$

where C_1 is a real constant whose value will be determined below. The function $\Psi(\zeta)$ denotes the branch for which

$$\lim \Psi(\zeta) e^{\zeta} = (-1)^{\varepsilon} C_1 \quad \text{as} \quad |\zeta| \to \infty \qquad (1.18)$$

is valid.

Rewriting Ohm's law in complex form

$$j(z) = j_x(x, y) - i j_y(x, y) =$$
$$= \frac{-\sigma}{1 + \omega_e \tau_e \omega_i \tau_i + i \omega_e \tau_e} \left(\frac{\partial \chi}{\partial x} - i \frac{\partial \chi}{\partial y}\right), \qquad (1.19)$$

and taking formulas (1.10), (1.15)–(1.19) into account, we find the required current distribution in a plane channel:

$$j(z) = j_x(x, y) - i j_y(x, y) =$$
$$= \frac{-C_1 \sigma (-1)^{\varepsilon}}{1 + \omega_e \tau_e \omega_i \tau_i + i \omega_e \tau_e} \exp \frac{\pi z}{h} \times$$
$$\times \left[\left(\exp \frac{\pi z}{h} - \exp - \frac{\pi l}{h}\right)\left(\exp \frac{\pi z}{h} + \exp \frac{\pi l}{h}\right)\right]^{-1/2+\varepsilon} \times$$
$$\times \left[\left(\exp \frac{\pi z}{h} - \exp \frac{\pi l}{h}\right)\left(\exp \frac{\pi z}{h} + \exp - \frac{\pi l}{h}\right)\right]^{-1/2-\varepsilon}. \qquad (1.20)$$

The formulas which have been obtained enable us to calculate the integral characteristics of the generator.

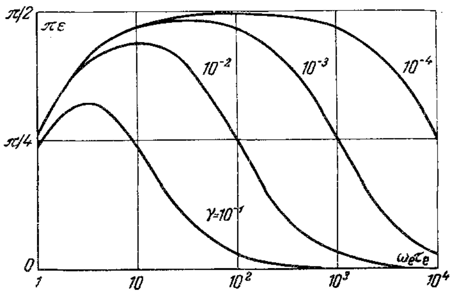

Fig. 4

The total current I, flowing through the generator load, is given by the formula

$$I = s \int_a^b j_y(x)\, dx = \frac{C_1 s \sigma (1 + \omega_e \tau_e \omega_i \tau_i) \Delta_1(l, h, \varepsilon)}{(1 + \omega_e \tau_e \omega_i \tau_i)^2 + \omega_e^2 \tau_e^2}, \qquad (1.21)$$

$$\Delta_1(l, h, \varepsilon) = \int_{-\pi l/h}^{\pi l/h} [(e^x - e^{-\pi l/h})(e^x + e^{\pi l/h})]^{-1/2+\varepsilon} \times$$
$$\times [(e^{\pi l/h} - e^x)(e^x + e^{-\pi l/h})]^{-1/2-\varepsilon} e^x dx$$
$$(-\pi l/h \leqslant x \leqslant \pi l/h), \qquad (1.22)$$

where s is the width of the electrodes (the channel height).

The internal voltage drop between the electrodes is found from the formulas

$$\left(\int_{-\infty}^{a} + \int_{b}^{\infty}\right)\left(\frac{\partial \varphi}{\partial x} + \frac{1}{c} H \frac{\partial \psi}{\partial x}\right)_{y=0} dx = \mathscr{E} - 2\varphi_e,$$

$$\mathscr{E} = \frac{1}{c} H \left(\int_{-\infty}^{a} + \int_{b}^{\infty}\right)\left(\frac{\partial \psi}{\partial x}\right)_{y=0} dx. \qquad (1.23)$$

Here $\mathscr{E}$ is the electromotive force and $2\varphi_e$ is the voltage between the electrodes.

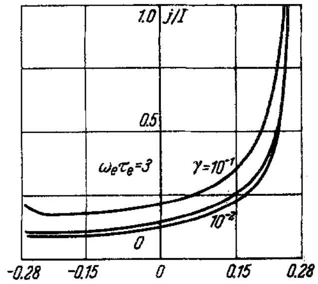

Fig. 5

If formula (1.20) is employed, the first relation of (1.23) may be reduced to the form

$$\mathscr{E} - 2\varphi_e = C_1 (1 + \omega_e \tau_e \omega_i \tau_i) [\Delta_2(l, h, \varepsilon) +$$
$$+ \Delta_3(l, h, \varepsilon)] [(1 + \omega_e \tau_e \omega_i \tau_i)^2 + \omega_e^2 \tau_e^2]^{-1/2}, \qquad (1.24)$$

$$\Delta_2(l, h, \varepsilon) = \int_{\pi l/h}^{\infty} [(e^x - e^{-\pi l/h})(e^x + e^{\pi l/h})]^{-1/2+\varepsilon} \times$$
$$\times [(e^x - e^{\pi l/h})(e^x + e^{-\pi l/h})]^{-1/2-\varepsilon} e^x dx$$
$$(x \geqslant \pi l/h),$$

$$\Delta_3(l, h, \varepsilon) = \int_{-\infty}^{-\pi l/h} [(e^{-\pi l/h} - e^x)(e^x + e^{\pi l/h})]^{-1/2+\varepsilon} \times$$
$$\times [(e^{\pi l/h} - e^x)(e^x + e^{-\pi l/h})]^{-1/2-\varepsilon} e^x dx$$
$$(x \leqslant -\pi l/h). \qquad (1.25)$$

The value of the constant C_1 is obtained from the equation

$$\mathscr{E} - 2\varphi_e = I\Omega_b, \quad \text{or} \quad \mathscr{E} - I\Omega_l = I\Omega_b. \qquad (1.26)$$

Here Ω_b is the internal resistance of the generator, and Ω_l is the generator load. Substituting the value

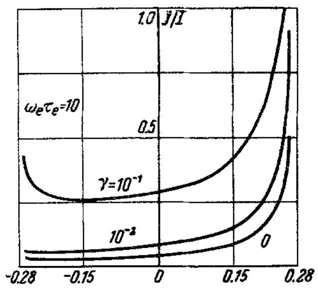

Fig. 6

of the current I determined from formula (1.21) into (1.26), we have

$$C_1 = \frac{\mathscr{E} [(1 + \omega_e \tau_e \omega_i \tau_i)^2 + \omega_e^2 \tau_e^2]}{\sigma s \Delta_1(l, h, \varepsilon)(1 + \omega_e \tau_e \omega_i \tau_i)(\Omega_b + \Omega_l)}. \qquad (1.27)$$

The internal resistance of the generator is given by the formula

$$\Omega_b = \frac{\mathscr{E} - 2\varphi_e}{I} =$$

$$= \frac{\Delta_2(l, h, \varepsilon) + \Delta_3(l, h, \varepsilon)}{\sigma s \Delta_1(l, h, \varepsilon)} [(1 + \omega_e\tau_e\omega_i\tau_i)^2 + \omega_e^2\tau_e^2]^{1/2}. \quad (1.28)$$

When the constant C_1 and the internal resistance Ω_b have been found, it is a simple matter to calculate the other integral characteristics of the generator

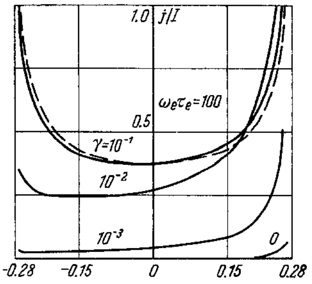

Fig. 7

[4–6]: the electrical power $N = 2\varphi_e I$ and the Joule dissipation in the channel $Q = I\mathscr{E} - N$. These formulas can be written in a simple and clear form if we introduce the load parameter

$$q = \frac{2\varphi_e}{\mathscr{E}} = \frac{\Omega_l}{\Omega_l + \Omega_b} \qquad (0 \leqslant q \leqslant 1). \qquad (1.29)$$

We then have

$$I = (1 - q)\frac{\mathscr{E}}{\Omega_b}, \quad N = q(1 - q)\frac{\mathscr{E}^2}{\Omega_b},$$

$$Q = (1 - q)^2 \frac{\mathscr{E}^2}{\Omega_b}. \qquad (1.30)$$

§2. It is clear that the electrical characteristics of a magnetohydrodynamic generator are basically determined by the value of the internal resistance Ω_b (for fixed $\mathscr{E}$ and q). According to (1.28), Ω_b in the general case depends on the geometric dimensions of the generator (l, h, and s), on the physical properties of the conducting medium in the channel, and on the strength of the external magnetic field H. Theoretically Ω_b may vary from zero to infinity, and the generator characteristics vary accordingly.

As mentioned above, the boundary-value problem (1.5), (1.7)–(1.9) can be solved by other methods. For example, the strip can first be transformed to a half-plane, and then to the interior of a parallelogram by means of a Schwarz-Christoffel integral [4–8]. Moreover, if we introduce an effective conductivity σ_* and the effective Hall parameter $\omega_*\tau_*$ as given by

$$\sigma_* = \frac{\sigma}{1 + \omega_e\tau_e\omega_i\tau_i}, \qquad \omega_*\tau_* = \frac{\omega_e\tau_e}{1 + \omega_e\tau_e\omega_i\tau_i} \qquad (2.1)$$

and set them in the inital equations (1.6) and in boundary conditions (1.7), we obtain the boundary-value problem solved in [4,6,8]. We can thus use the results of these papers to analyze the characteristics of magnetohydrodynamic energy converters when "slipping" of ions occurs.

It is convenient to use approximate formulas to find the internal resistance of the generator. One such formula can be derived from (1.28) by calculating the improper integrals (1.22) and (1.25) according to the method of residues

$$\Omega_b = \frac{1}{\sigma s}[(1 + \gamma\omega_e^2\tau_e^2)^2 +$$

$$+ \omega_e^2\tau_e^2]^{1/2}\left[1 + \frac{1 - 2\varepsilon}{1 + 2\varepsilon}\left(2\,\mathrm{th}\frac{\pi l}{h}\right)^{-2\varepsilon}\right] \times$$

$$\times\left[\left(1 - \exp -\frac{\pi l}{h}\right)^{1/2 - \varepsilon} +\right.$$

$$\left. + \frac{1 - 2\varepsilon}{1 + 2\varepsilon}\left(2\,\mathrm{th}\frac{\pi l}{h}\right)^{-2\varepsilon}\left(\exp\frac{\pi l}{h} - 1\right)^{1/2 + \varepsilon}\right]^{-1},$$

$$\varepsilon = \frac{1}{\pi}\,\mathrm{arc\,tg}\,\frac{\omega_e\tau_e}{1 + \gamma\omega_e^2\tau_e^2} \quad \left(0 \leqslant \varepsilon < \frac{1}{2}\right), \quad \gamma = \frac{\omega_i\tau_i}{\omega_e\tau_e}. \quad (2.2)$$

The value of the coefficient γ is defined by the chemical composition of the conducting medium and its physical state. For the cases which are of most interest in practice, γ varies within the limits

$$\gamma = \omega_i\tau_i / \omega_e\tau_e \approx 10^{-1} - 10^{-4}.$$

If $\omega_e\tau_e = 0$, expression (2.2) reduces to the form

$$\Omega_b = \frac{1}{\sigma s}\frac{2\exp(-\pi l/2h)}{[1 + \exp(-\pi l/h)][1 - \exp(-\pi l/h)]^{1/2}}. \qquad (2.3)$$

According to (1.28) the exact expression corresponding to this case is

$$\Omega_b = \frac{1}{\sigma s}\frac{\Delta_2(l, h) + \Delta_3(l, h)}{\Delta_1(l, h)} = \frac{2}{\sigma s}\frac{K(k)}{K(k')}$$

$$\left(\begin{matrix} k = \exp(-2\pi l/h) \\ k' = \sqrt{1 - k^2} \end{matrix}\right), \qquad (2.4)$$

where K(k) is an elliptic integral of the first kind.

Curves of $\sigma s\Omega_b$ are given in Fig. 2 to illustrate how the internal resistance of the generator depends on the linear dimensions of the channel. The solid curve corresponds to the exact formula (2.4), and the dashed curve to the approximate formula (2.3). The approximate expressions (2.2) and (2.3) may be used for intermediate values of l/h.

In one case, when

$$\mathrm{sh}(\pi l/h) = 1 \quad \text{and} \quad l/h \approx 0.28, \qquad (2.5)$$

the integrals Δ_i (i = 1,2,3) in (1.28) reduce to beta-functions and the expression for the internal resistance assumes the very simple form

$$\Omega_b = \frac{1}{\sigma s}[(1 + \gamma\omega_e^2\tau_e^2)^2 + \omega_e^2\tau_e^2]^{1/2}. \qquad (2.6)$$

It is clear from this case that when the ions have spiral paths, the value of Ω_b increases as the parameter $\omega_e\tau_e$ increases, and the greater the value of the coefficient γ, the larger this rate of increase (Fig. 3).

When $\omega_i\tau_i = 0$ and $\omega_e\tau_e \gg 1$, Ω_b is an almost linear function of the parameter $\omega_e\tau_e$ for small values of l/h, and becomes a quadratic function of $\omega_e\tau_e$ for large values of l/h. This property of the internal resistance Ω_b can be explained physically by the fact that ideally conducting electrodes tend to short-circuit the Hall emf. The screening of the Hall emf naturally decreases for shorter electrodes (and in the limit case of point electrodes disappears entirely). Thus, segmented electrodes are sometimes used to improve the characteristics of magnetohydrodynamic energy converters for $\omega_e\tau_e \neq 0$. The situation is quite different when the ions as well as the electrons have spiral paths. An increase in the effective electrical conductivity $\sigma_* = \sigma/(1 + \gamma\omega_e^2\tau_e^2)$ is now also a contributing factor to the increase in Ω_b with increasing $\omega_e\tau_e$. In this case the internal resistance of the generator is a quadratic function for $\omega_e\tau_e \gg 1$ even when there is no Hall emf in the channel.

We also note that since the generator emf $\mathscr{E}$ ($\mathscr{E} = c^{-1}s^{-1}HQ_0$, where Q_0 is the rate of flow for the medium in the channel) is a linear function of the magnetic field, as $\omega_e\tau_e$ increases because of the magnetic field, when $\omega_i\tau_i \neq 0$ the generator power N has an absolute upper limit according to (1.30), which is determined by the ratio $\omega_i\tau_i/\omega_e\tau_e$.

It has already been noted that determining the integral characteristics of the magnetohydrodynamic generator (which, according to (1.29)–(1.30), is the same as finding the internal resistance) for arbitrary values of $\omega_e\tau_e$, $\omega_i\tau_i$, and l/h involves tedious calculations of the improper (but converging) integrals (1.22) and (1.25). However, it is possible in principle to determine the generator characteristics on a model, for which we may use a semiconducting slab situated in a magnetic field. For these purposes it is convenient to use a slab of finite dimensions, in the form of a round disk, for example, with electrodes connected to a source of electric current [9]. It is a simple matter to establish the correspondence between the geometric parameters of the channel and of the round semiconducting disk by means of conformal mapping (the appropriate transformation is given by the function $w = \mathrm{tg}^{1/2}\mathrm{ih}(1 - 4z/\pi)$. The over-all resistance of the slab is at the same time shown to correspond to the internal resistance of the generator. The resistances become identical when the physical constants of the semi-conducter and the conducting medium in the channel are the same.

§3. The distribution of electric current in the channel exhibits a marked dependence on the Hall parameter $\omega_e\tau_e$ and on the coefficient γ. When $\gamma = 0$ and $\omega_e\tau_e \neq 0$ (no "slipping" of ions) we know that the streamlines are distorted in the central zone of the channel owing to the anisotropy of conductivity in the medium, and concentrate mainly at the ends of the continuous electrodes as a result of the screening of the Hall emf by the ideal conductors. The quantitative characteristics of these phenomena are determined by the Hall angle $\pi\varepsilon = \mathrm{arc\,tg}\,\omega_e\tau_e$ $(0 \leq \pi\varepsilon < \pi/2)$, which is formed between the electric field-strength vector and the current-density vector.

The apperance of ion Larmor rotation leads to a decrease of the Hall angle. Curves for $\pi\varepsilon$ as a function of the Hall parameter $\omega_e\tau_e$, constructed from formula (1.17) for various values of γ, are given in Fig. 4. Inspection of the curves shows that when there is more "slipping" of the ions, the Hall angle $\pi\varepsilon$ decreases more rapidly as the parameter $\omega_e\tau_e$ increases. The angle $\pi\varepsilon$ assumes its maximum values when the condition $\omega_e\tau_e = \gamma^{-1/2}$ is satisfied. As $\pi\varepsilon$ decreases, the anisotropy of conductivity and the Hall emf in the channel disappear.

We shall now consider how the normal component of the electric field at the electrode is distributed in this case. One particular case of current distribution for $\omega_e\tau_e \neq 0$ and $\omega_i\tau_i \neq 0$ was treated in [10] by the method of finite differences. Let us assume that $l/h \approx 0.28$. We then have from (1.20), after some transformations,

$$j_y(x) = \frac{C_1\sigma(1+\gamma\omega_e^2\tau_e^2)}{2[(1+\gamma\omega_e^2\tau_e^2)^2+\omega_e^2\tau_e^2]}\frac{(1+\mathrm{sh}\,x)^{-1/2+\varepsilon}}{(1-\mathrm{sh}\,x)^{1/2+\varepsilon}}$$
$$(-0.88 \leqslant x \leqslant 0.88). \quad (3.1)$$

Determining the constant from (1.21) and substituting it into (3.1), we obtain

$$j_y(x) = \frac{2^{3/2}I}{B(1/4[1+2\varepsilon],\, 1/4[1-2\varepsilon])}\frac{(1+\mathrm{sh}\,x)^{-1/2+\varepsilon}}{(1-\mathrm{sh}\,x)^{1/2+\varepsilon}}$$
$$(-0.88 \leqslant x \leqslant 0.88), \quad (3.2)$$

where B(p, q) is a beta-function; I is the total current flowing through the electrode.

Curves of j/I constructed from this formula are given in Figs. 5, 6, and 7, respectively, for three values of the Hall parameter $\omega_e\tau_e$ = 3, 10, and 100 for various γ.

As $\omega_e\tau_e$ and γ increase, the distribution of the normal current component tends to the case $\omega_e\tau_e = 0$ (dashed curve in Fig. 7).

When $\pi\varepsilon = 0$ and $\omega_e\tau_e$ is increased, the internal resistance of the generator increases. This follows from (1.28)

$$\Omega_b = \frac{1+\gamma\omega_e^2\tau_e^2}{\sigma s}\frac{2K(k)}{K(k')} \qquad \begin{pmatrix} k = \exp(-2\pi l/h) \\ k' = \sqrt{1-k^2} \end{pmatrix}. \quad (3.3)$$

Here, as in (2.5), K(k) is a complete elliptic integral of the first kind. The numerical factor 2 in (3.3) can be eliminated by the transformation formula for elliptic integrals

$$K(k') = \frac{2}{1+k'}K\left(\frac{1-k'}{1+k'}\right), \quad K(k) = \frac{1}{1+k}K\left(\frac{2\sqrt{k}}{1+k}\right).$$

After transformation we obtain

$$\Omega_b = \frac{1+\gamma\omega_e^2\tau_e^2}{\sigma s}\frac{K(k_1)}{K(k_1')} \qquad \begin{pmatrix} k_1 = \mathrm{th}\,\pi l/h \\ k_1' = \sqrt{1-k_1^2} \end{pmatrix}.$$

§4. The electric field in a channel with many sections can be calculated easily by generalizing the formulas of §1.

Let a finite number of electrodes be situated on the channel walls, without specifying their dimensions or connecting circuit. The initial statements and the formulation of the problem remain as before. The field in such a channel was calculated by I. M. Tolmach and N. N. Yasnitskaya [11], when there is no ion Larmor rotation.

The current distribution in the channel when ion "slipping" is taken into account is given by the following formula:

$$j(z) = j_x(x, y) - ij_y(x, y) =$$
$$= \frac{-(-1)^\varepsilon\sigma}{1+\gamma\omega_e^2\tau_e^2+i\omega_e\tau_e}P\left(\exp\frac{\pi z}{h}\right)\times$$
$$\times\prod_{k=1}^{p}\left[\left(\exp\frac{\pi z}{h}-\exp a_k\right)\left(\exp\frac{\pi z}{h}-\exp b_k'\right)\right]^{-1/2+\varepsilon}\times$$
$$\times\left[\left(\exp\frac{\pi z}{h}-\exp b_k\right)\left(\exp\frac{\pi z}{h}-\exp a_k'\right)\right]^{-1/2-\varepsilon}, \quad (4.1)$$

$$P\left(\exp\frac{\pi z}{h}\right) = C_{2p-1}\exp\left((2p-1)\frac{\pi z}{h}\right)+$$
$$+C_{2p-2}\exp\left((2p-2)\frac{\pi z}{h}\right)+\ldots+C_1\exp\frac{\pi z}{h},$$

$$\varepsilon = \frac{1}{\pi}\mathrm{arc\,tg}\,\frac{\omega_e\tau_e}{1+\gamma\omega_e^2\tau_e^2}, \qquad 0 \leqslant \varepsilon < \frac{1}{2}. \quad (4.2)$$

Here $a_k b_k$ and $a_k' b_k'$ ($k = 1, \ldots, 2p - 1$) are the end points of the electrodes on the two channel walls. The real constants C_k ($k = 1, \ldots, 2p - 1$) in formulas (4.1) and (4.2) are still unknown, and it is these which allow the solution to be fitted to a particular concrete case. To find these constants we must take into account the circuit which connects the electrodes to the load.

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REFERENCES

1. A. G. Kulikovskii and G. A. Lyubimov, Magnitnaya Gidrodinamika [in Russian], Fizmatgiz, 1962.

2. L. I. Chibrikova, "The Riemann boundary-value problem for automorphic functions," Uch. zap. Kazansk. un-ta, vol. 116, no. 4, pp. 59–109, 1956.

3. F. D. Gakhov, Boundary-Value Problems [in Russian], Fizmatgiz, 1963.

4. A. B. Vatazhin, "Magnetohydrodynamic flow in a flat channel with finite electrodes," Izv. AN SSSR, OTN, Mekhanika i Mashinostroenie, no. 1, pp. 52–58, 1962.

5. A. B. Vatazhin, "The electric field in magnetohydrodynamic channels with anisotropic conductivity." Fifth Riga Conference on Magnetohydrodynamics, Riga, no. 14, pp. 37–77, 1966.

6. F. Shultz-Grunow and D. L. Danzel, "Calculation of the electric characteristics of an MHD generator with finite electrodes by conformal mapping under full consideration of tensor conductivity and a velocity profile." Internat. Sympos. Magnetohydrodinam. Generation, Paris, vol. 2, 1964.

7. A. B. Vatazhin, "Toward the solution of certain problems in magnetohydrodynamics," PMM, vol. 25, no. 5, pp. 965–968, 1961.

8. A. B. Vatazhin, "Some two-dimensional problems on current distribution in an electrically conducting medium moving along a channel in a magnetic field," PMTF, no. 2, pp. 39–54, 1963.

9. P. I. Baranskii and Yu. P. Emets, "The electric field in a round semiconducting slab situated in a magnetic field," PMTF [Journal of Applied Mechanics and Technical Physics], no. 5, pp. 64–72, 1966.

10. T. R. Denison and R. W. Ziemer, "Investigation of the phenomena in crossed-field plasma accelerations," Phys.-Chem. Diagnostics Plasmas, Northwest Univ. Press, Evanston, p. 201–232, 1963.

11. I. M. Tolmach and N. N. Yasnitskaya, "The Hall effect in a channel with segmented electrodes," Izv. AN SSSR, Energetika i transport, no. 5, pp. 91–104, 1965.

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